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Renormalization group and decoupling in curved space

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Abstract: It is well known that the renormalization group equations depend on the scale where they are applied. This phenomenon is especially relevant for the massive fields in curved space, because the decoupling effects may be responsible for important cosmological applications like the graceful exit from the inflation and low-energy quantum dynamics of the cosmological constant. We investigate, using both covariant and non-covariant methods of calculations and mass-dependent renormalization scheme, the vacuum quantum effects of a massive scalar field in curved space-time. In the higher derivative sector we arrive at the explicit form of decoupling and obtain the β-functions in both UV and IR regimes as the limits of general expressions. For the cosmological and Newton constants the corresponding β-functions are not accessible in the perturbative regime and in particular the form of decoupling remains unclear.

Keywords: Renormalization Group, Physics of the Early Universe.

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Contents

1. Introduction

The renormalization group methods represent one of the most universal tools in the Quantum Field Theory and beyond: from string theory to lattice QCD and statistical mechanics. In the last decades, there was a growing interest to the renormalization group for quantum fields in curved space-time, especially in relation to the anomalies and cosmological applications. The renormalization group and related phenomena of the decoupling of the massive quantum matter fields in an external gravitational field are especially relevant for such phenomena as the graceful exit from the anomaly-induced inflation [1, 2] and the possible low-energy quantum dynamics of the cosmological constant [3]. The relevance of these applications (which may help in better understanding of the coincidence puzzle for the ΛCDM cosmological model and also may lead to a natural theory of inflation) requires a detailed quantitative analysis of the physical aspects of the renormalization group. In this paper we are going to start this investigation.

The standard approach to the renormalization group in an external gravitational field has been formulated in [4, 5] (see also [6, 7]) on the basis of the Minimal Subtraction (MS) renormalization scheme or the modified Minimal Subtraction scheme $\overline{\mathrm{MS}}$. In the framework of the $\overline{\mathrm{MS}}$ scheme, the renormalization of quantum matter in curved space-time is well understood [8]. The principal results are quite simple and look as follows. Starting from the renormalizable QFT in the flat space-time, one can always construct the renormalizable QFT in curved space-time. In particular, if we treat the gravitational field as a perturbation over the flat background, then the presence of an external gravity does not increase the superficial degree of divergence for a given Feynman diagram. Taking the covariance and locality considerations into account, one can easily determine the complete set of possible divergent structures and identify the corresponding necessary counterterms.

In general, the action of the renormalizable theory includes the covariant generalization of the action in flat space plus a set of curvature-dependent local terms, which are necessary for the renormalizability. In particular, one has to introduce a special action of vacuum (external gravity[1]) which has the form

$$S_{vacuum} = S_{EH} + S_{HD}\,. \tag{1.1}$$

Here the first term is the Einstein-Hilbert action with cosmological constant. [2]

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x g^{1/2}(R+2\Lambda)\,. \tag{1.2}$$

The second action contains necessary higher derivative terms

$$S_{HD} = \int d^4x g^{1/2}\left\{a_1 C^2 + a_2 R^2 + a_3 E + a_4 \nabla^2 R\right\}, \tag{1.3}$$

where

$$C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + \frac{1}{3}R^2 \tag{1.4}$$

is the square of the Weyl tensor and

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 \tag{1.5}$$

is the integrand of the Gauss-Bonnet topological invariant. $a_1, \dots, a_4$ (and also G and Λ in (1.2)) are the parameters of the vacuum action. In the present paper we shall focus our attention on the renormalization of the vacuum parameters a_i, G, Λ and on the quantum corrections to the vacuum action (1.1).

For the interacting fields there is also renormalization in the matter fields sector. Let us remark that the renormalization of those terms that have direct analogs in flat space is exactly the same as in the flat space [8]. In addition, there is also the renormalization of the nonminimal parameter ξ in the scalar action

$$S_s = \frac{1}{2}\int d^4x g^{1/2}\left\{g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2 + \xi R\varphi^2\right\}. \tag{1.6}$$

The renormalization of ξ has many interesting features (see, e.g. [8, 9, 10]) but is beyond the scope of the present paper because, in the cosmological framework, the effect of this renormalization is less important than the renormalization of vacuum terms.

The vacuum divergences of the theory (1.6) can be easily calculated using the Schwinger-DeWitt method and dimensional regularization

$$\bar{\Gamma}^{(1)}_{div} = -\frac{(\mu^2)^{\omega-2}}{2(4\pi)^2(\omega-2)}\int d^{2\omega}x\, g^{1/2}\left\{\frac{1}{180}\left(R^2_{\mu\nu\alpha\beta} - R^2_{\alpha\beta} - \nabla^2 R\right) - \frac{1}{6}\left(\xi - \frac{1}{6}\right)\nabla^2 R + \right.$$
$$\left. + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R^2 + m^2\left(\xi - \frac{1}{6}\right)R + \frac{1}{2}m^4\right\}, \tag{1.7}$$

[1]Let us emphasize that gravity itself is not quantized in this paper, so we are confined to the so-called semiclassical approach.

[2]Our notations are euclidean metric $\eta_{\mu\nu} = diag(++++)$ and the definition of the curvature tensor $R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \dots$.

where ω is the parameter of the dimensional regularization (2ω is the space-time dimension) and μ is a dimensional parameter which is introduced in order to compensate the change of the dimension. The renormalization of the vacuum action in the $\overline{\mathrm{MS}}$ scheme proceeds as follows. One has to introduce the local counterterm ΔS and then compare the bare $S_0[\Lambda^{(0)}, G^{(0)}, a_i^{(0)}]$ and renormalized $S_R[\Lambda, G, a_i] = S + \Delta S$ actions. This leads to the standard renormalization relations (see, e.g. [8]). In particular,

$$-\frac{\Lambda^{(0)}}{8\pi G^{(0)}} = \left(\mu^2\right)^{\omega-2}\left[-\frac{\Lambda}{8\pi G} + \frac{m^4}{4(4\pi)^2(\omega-2)}\right]. \tag{1.8}$$

Taking into account the fact that $\Lambda^{(0)}$ does not depend on μ, we arrive at the usual (see, e.g. [11]) renormalization group equation for the cosmological constant

$$\mu\frac{d}{d\mu}\left(\frac{\Lambda}{8\pi G}\right) = \beta_\Lambda\left(\overline{MS}\right) = \frac{m^4}{2(4\pi)^2}\,, \tag{1.9}$$

where we have set $\omega = 2$. Using similar calculations, we arrive at the renormalization group equations for other vacuum parameters (see, e.g. [4])

$$\begin{aligned} \mu\frac{d}{d\mu}\left(-\frac{1}{16\pi G}\right) &= \beta_R(\overline{MS}) = \frac{m^2}{(4\pi)^2}\left(\xi - \frac{1}{6}\right), \\ \mu\frac{da_1}{d\mu} &= \beta_1(\overline{MS}) = -\frac{1}{120(4\pi)^2}\,, \\ \mu\frac{da_2}{d\mu} &= \beta_2(\overline{MS}) = -\frac{1}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right)^2. \end{aligned} \tag{1.10}$$

Let us emphasize that these equations were obtained within covariant formalism and without imposing restrictions on the external metric $g_{\mu\nu}$.

The interpretation of the renormalization group equations in curved space-time requires a special attention. Of course, this is a usual situation for the $\overline{\mathrm{MS}}$ scheme, where even in flat space one has to make a special effort in order to interpret the parameter μ. Since μ is an auxiliary parameter introduced in the dimensional regularization, to get physical results, one should trade it for some physical quantity. This procedure may lead to ambiguities, which can be resolved in the framework of the more physical mass-dependent scheme of renormalization. For the usual flat-space theories, say in QED or Yang-Mills, the interpretation of μ can be achieved as follows. One can compare the renormalization group in the $\overline{\mathrm{MS}}$ scheme and in a mass-dependent scheme, which must coincide in the high energy limit. In this way, μ acquires a natural interpretation. Depending on the situation this may be the energy of scattering of quanta, VEV of the scalar field or temperature etc.

On the other hand, at low energies, the physically correct result requires the use of a mass-dependent scheme. But, in the case of the vacuum renormalization in a curved space-time, the notion of energy for the gravitational quanta is not well defined, and the relation between $\overline{\mathrm{MS}}$ and mass-dependent scheme is a priori unclear. At the same time, there is a considerable interest in implementing ideas of effective Quantum Field Theory into the gravitational framework: both for the classical gravitational background and effective quantum gravity [12]. To this end, we need to achieve a quantitative description of how the decoupling of the heavy particles in the external gravitational field occurs.

Let us briefly remind how decoupling proceeds in flat space in a mass-dependent scheme by using the QED example (see, e.g. [13]). The 1-loop vacuum polarization in the dimensional regularization is equal to

$$\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\frac{1}{6(\omega-2)} - \frac{\gamma}{6} - \int_0^1 dx x(1-x)\ln\frac{m_e^2 + p^2 x(1-x)}{4\pi\mu^2}\right], \tag{1.11}$$

where p is an external momentum, m is fermion mass, γ is Euler's constant, and μ is the dimensional parameter of the regularization. By subtracting in the $\overline{MS}$ scheme the $\frac{1}{\omega-2}$-pole and the γ-term, one find the following contribution:

$$-\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\int_0^1 dx x(1-x)\ln\frac{m_e^2 + p^2 x(1-x)}{4\pi\mu^2}\,. \tag{1.12}$$

The β-function in the $\overline{\mathrm{MS}}$ scheme is obtained by calculating the derivative $(e/2)\mu d/d\mu$ acting on the coefficient of the operator $(p_\mu p_\nu - p^2 g_{\mu\nu})$. The result is well known

$$\beta_e\left(\overline{\mathrm{MS}}\right) = \frac{e^3}{12\pi^2}. \tag{1.13}$$

Thus, the β-function in the $\overline{MS}$ scheme is a constant independent of the fermion mass and μ. To calculate the β-function in a mass-dependent scheme such as, e.g., the momentum space subtraction scheme, one has to subtract the value of the graph at a point $p^2 = M^2$ and calculate the derivative $(e/2)Md/dM$ acting on the coefficient of the $(p_\mu p_\nu - p^2 g_{\mu\nu})$ projector. As a result, we obtain

$$\beta_e = \frac{e^3}{2\pi^2}\int_0^1 dx\, x(1-x)\frac{M^2 x(1-x)}{m_e^2 + M^2 x(1-x)}. \tag{1.14}$$

It is easy to verify that when the renormalization point is much larger than the fermion mass, the β-function obtained in the mass-dependent scheme coincides with the $\overline{MS}$ β-function (1.13). As the renormalization point passes through m_e, massive fermion decouples and, for $M \ll m_e$, its contribution to the β-function vanishes as

$$\beta_e(M \ll \mu_e) \approx \frac{e^3}{60\pi^2}\frac{M^2}{m_e^2}. \tag{1.15}$$

Thus, the $\overline{MS}$ β-function gives wrong result at energies less than m_e. Decoupling of heavy particles is implemented in the $\overline{MS}$ scheme by putting, artificially, the β-function to zero for $\mu < m$. This can be called "hard decoupling" or "sharp cut-off" approximation. Indeed, this is just an approximation to the real picture of decoupling, which may be essentially scheme-dependent. In particular, the example of QED shows that, in principle, the decoupling in gravity can proceed in a soft manner such that the β-function is continuous at the scale m.

The purpose of this paper is to perform, for the first time, the derivation of the renormalization group equations for the parameters Λ, G, a_1 and a_2 in a simple mass-dependent scheme. We consider a massive non-minimally coupled scalar theory, but the results can be also generalized for massive fermions and vectors. In section 2 we present the derivation of the polarization operator of gravitons using dimensional regularization. The table of cumbersome integrals are exposed in the appendix A. Although we will later confirm our principal conclusions by using covariant heat kernel calculations, we believe that the linearized gravity calculations are useful. These calculations are performed on the flat space background and, hence, they are maximally close to usual perturbative quantum field theory calculations, where we have a good intuition of how the decoupling of massive particles works. In section 3, the polarization operator of section 2 is verified using the previously known expressions for the summation of the Schwinger-DeWitt series [14, 15]. In section 4, the renormalization group equations for the vacuum parameters are derived and the comparison with similar equations in the $\overline{\rm MS}$ scheme is given. Another calculation which is aimed to clarify the relation between the $\overline{\rm MS}$ and mass-dependent scheme is presented in the appendix B, where we apply the covariant cut-off regularization of the proper-time integral representation for the effective action. Unfortunately, in this regularization scheme one can recover the reliable form of the β-functions only at high energy limit, while the small cut-off limit is inconsistent. In section 5 we discuss the results of the calculations in the mass-dependent scheme. In particular, this opportunity is used to discuss the form of the effective action which could correspond to the running of the cosmological and inverse Newton constants at low energies. After all, we draw our conclusions.

In order to avoid the confusion, let us fix the notations for the distinct types of the β-functions. We shall denote, in this paper, the $\overline{\rm MS}$-scheme β-function for the effective charge C as $\beta_C(\overline{\rm MS})$ and the β-function for the same object, derived in a mass-dependent scheme as β_C. The high energy limit of this β-function will be denoted as β_C^{UV} and the low-energy limit as β_C^{IR}. Of course, we expect that the correctly defined β-function would satisfy the relation

$$\beta_C^{UV} = \beta_C\left(\overline{\rm MS}\right) + \mathcal{O}\left(\frac{m^2}{p^2}\right).$$

2. The non-covariant perturbative calculation

The renormalization group for the parameters of the vacuum action can be established through the perturbative calculation of the quantum corrections to the gravitational 2-point function. Therefore, in this section we shall derive the contributions of the loop of massive particle to the propagator of the gravitational perturbations $h_{\mu\nu}$ on the flat background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{2.1}$$

The calculations will be performed in the euclidean space. The diagrams which contribute to the gravitational polarization operators are presented at figure 1.

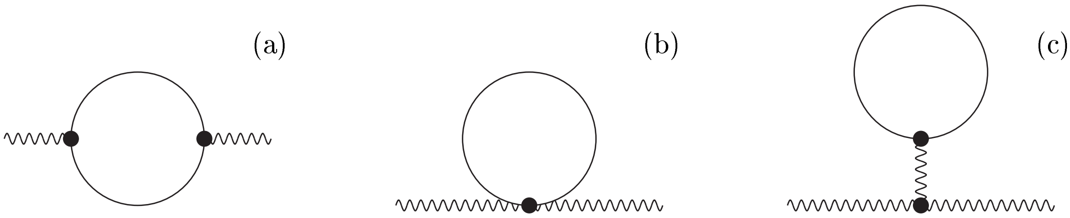

Figure 1: The 1-loop diagrams with the loop of massive scalar and two external gravitational lines. The straight lines correspond to the massive scalar and wavy lines to the external field $h_{\mu\nu}$.

Figure 2: Two relevant vertices of the gravity-scalar interactions.

The 3-point and 4-point vertices (see figure 2) have the form

$$V^{\alpha\beta}(k,p,q) \;=\; (2\pi)^4\left[\frac{1}{2}\left(\eta^{\alpha\beta}p\cdot q - q^\alpha p^\beta - p^\alpha q^\beta\right) + \xi\left(k^\alpha k^\beta - k^2\eta^{\alpha\beta}\right) - \frac{m^2}{2}\eta^{\alpha\beta}\right] \tag{2.2}$$

$$\begin{aligned}
V^{\mu\nu,\alpha\beta}(k,p,q,l) \;=\; (2\pi)^4\Bigg\{&\delta^{\mu\nu,\alpha\beta}\left[\xi\left(k^2+l^2+3k\cdot l\right) - \frac{1}{2}p\cdot q\right] + \\
&+\eta^{\mu\nu}\eta^{\alpha\beta}\left[\frac{1}{4}p\cdot q - \frac{\xi}{2}\left(k^2+l^2+k\cdot l\right)\right] + \\
&+\eta^{\mu\nu}\left[-\frac{1}{2}p^\alpha q^\beta + \xi\left(k^\alpha k^\beta + \frac{1}{2}l^\alpha l^\beta + k^\alpha l^\beta\right)\right] + \\
&+\eta^{\alpha\beta}\left[-\frac{1}{2}p^\mu q^\nu + \xi\left(l^\mu l^\nu + \frac{1}{2}k^\mu k^\nu + k^\mu l^\nu\right)\right] + \\
&+\eta^{\mu\alpha}\left[p^\nu q^\beta + q^\nu p^\beta - 2\xi\left(k^\nu k^\beta + l^\nu l^\beta\right) - 2\xi k^\nu l^\beta - \xi l^\nu k^\beta\right]\Bigg\}\,,
\end{aligned} \tag{2.3}$$

where, for the sake of compactness, we do not maintain the symmetry inside the couples of indices $(\alpha\beta)$ and $(\mu\nu)$.

The first observation is that the diagram ($1c$) does not contribute to the renormalization group because we are dealing with the semiclassical approximation. The object of interest is the effective action of the external metric, and its expansion into series in $h_{\mu\nu}$ does not lead to the diagram ($1c$). For this reason we have to consider only ($1a$) and ($1b$) diagrams. Let us start from ($1a$). The lengthy calculations in the dimensional regularization give the following expression for the polarization operator

$$\begin{aligned}
\Pi_{1a}^{\mu\nu,\alpha\beta}(p) \;=\; &-\frac{1}{16}\eta^{\mu\nu}\eta^{\alpha\beta}\left(p^4 I_1 - 2p^4 I_2 + 6p^2 I_3 + 2p^4 I_4 + 8 I_9\right) - \\
&-\frac{1}{2}p^\mu p^\nu p^\alpha p^\beta (I_7 - 2I_5 + I_4) -
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{8}\left(\eta^{\mu\alpha}p^\nu p^\beta+\eta^{\mu\beta}p^\nu p^\alpha+\eta^{\nu\alpha}p^\mu p^\beta+\eta^{\nu\beta}p^\mu p^\alpha\right)(4I_8-4I_6+I_3)-\delta^{\mu\nu,\alpha\beta}I_9-\\
&-\frac{1}{8}\left(\eta^{\mu\nu}p^\alpha p^\beta+\eta^{\alpha\beta}p^\mu p^\nu\right)(p^2I_4-p^2I_2+4I_8-4I_6)-\\
&-\frac{\xi^2}{2}I_1\left(p^\mu p^\nu-p^2\eta^{\mu\nu}\right)\left(p^\alpha p^\beta-p^2\eta^{\alpha\beta}\right)-\\
&-\frac{\xi}{8}\left(p^\mu p^\nu-p^2\eta^{\mu\nu}\right)\eta^{\alpha\beta}\left(p^2I_1+4I_3\right)-\frac{\xi}{8}\left(p^\alpha p^\beta-p^2\eta^{\alpha\beta}\right)\eta^{\mu\nu}\left(p^2I_1+4I_3\right)+\\
&+\frac{\xi}{4}\left(p^\mu p^\nu-p^2\eta^{\mu\nu}\right)p^\alpha p^\beta\left(I_1-2I_4\right)+\frac{\xi}{4}\left(p^\alpha p^\beta-p^2\eta^{\alpha\beta}\right)p^\alpha p^\beta(I_1-2I_4)+\\
&+\frac{\Gamma(1-\omega)}{2(4\pi)^\omega}(m^2)^{\omega-1}\left[\frac{\xi}{2}\left(p^\mu p^\nu\eta^{\alpha\beta}+p^\alpha p^\beta\eta^{\mu\nu}\right)+\left(\frac{1}{4}-\xi\right)p^2\eta^{\alpha\beta}\eta^{\mu\nu}\right]+\\
&+\frac{\Gamma(-\omega)}{4(4\pi)^\omega}(m^2)^\omega\eta^{\mu\nu}\eta^{\alpha\beta}\,, \qquad (2.4)
\end{aligned}
$$

where the integrals $I_1, \ldots, I_9$ are defined in the appendix A.

The second diagram is simpler and the corresponding expression is

$$
\begin{aligned}
\Pi_{1b}^{\mu\nu,\alpha\beta}(p) = {} & \frac{\xi}{4}\frac{\Gamma(1-\omega)}{(4\pi)^\omega}(m^2)^{\omega-1}\Big[p^2\delta^{\mu\nu,\alpha\beta}-p^2\eta^{\mu\nu}\eta^{\alpha\beta}+\left(p^\mu p^\nu\eta^{\alpha\beta}+p^\alpha p^\beta\eta^{\mu\nu}\right)-\\
&-\frac{1}{2}\left(\eta^{\mu\alpha}p^\nu p^\beta+\eta^{\mu\beta}p^\nu p^\alpha+\eta^{\nu\alpha}p^\mu p^\beta+\eta^{\nu\beta}p^\mu p^\alpha\right)\Big]+\\
&+\frac{\Gamma(-\omega)(\omega-2)}{8(4\pi)^\omega}(m^2)^\omega\left(2\delta^{\mu\nu,\alpha\beta}-\eta^{\mu\nu}\eta^{\alpha\beta}\right). \qquad (2.5)
\end{aligned}
$$

In order to apply the mass-dependent renormalization scheme, we need both divergent and finite parts of the polarization operator. Let us start from divergencies and establish the correspondence with the covariant expression (1.7).

For this end, since we are working with the polarization operator, the terms in the vacuum action (1.1) must be expanded up to the second order in $h_{\mu\nu}$. As far as the Gauss-Bonnet term does not contribute to the propagator, we can replace the $C^2_{\mu\nu\alpha\beta}$-term by the expression $2W$, where $W = R^2_{\mu\nu} - \frac{1}{3}R^2$. Then the relevant bilinear expressions have the form

$$
\int d^4x g^{1/2} = \int d^4x h^{\mu\nu}\Big(\frac{1}{8}\eta_{\mu\nu}\eta_{\alpha\beta}-\frac{1}{4}\delta_{\mu\nu,\alpha\beta}\Big)h^{\alpha\beta}+\cdots\,, \qquad (2.6)
$$

$$
\begin{aligned}
\int d^4x g^{1/2}R = {} & \int d^4x h^{\mu\nu}\Big[\frac{1}{4}\delta_{\mu\nu,\alpha\beta}\partial^2-\frac{1}{4}\eta_{\mu\nu}\eta_{\alpha\beta}\partial^2+\\
&+\frac{1}{4}(\eta_{\mu\nu}\partial_\alpha\partial_\beta+\eta_{\alpha\beta}\partial_\mu\partial_\nu)-\frac{1}{2}\eta_{\mu\alpha}\partial_\nu\partial_\beta)\Big]h^{\alpha\beta}+\cdots\,, \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
\int d^4x g^{1/2}R^2 = {} & \int d^4x h^{\mu\nu}\Big[\partial_\mu\partial_\nu\partial_\alpha\partial_\beta+\eta_{\mu\nu}\eta_{\alpha\beta}\partial^2-\\
&-(\eta_{\mu\nu}\partial_\alpha\partial_\beta+\eta_{\alpha\beta}\partial_\mu\partial_\nu)\partial^2\Big]h^{\alpha\beta}+\cdots\,, \qquad (2.8)
\end{aligned}
$$

$$
\begin{aligned}
\int d^4x g^{1/2}W = {} & \int d^4x h^{\mu\nu}\Big[\frac{1}{4}\delta_{\mu\nu,\alpha\beta}\partial^4-\frac{1}{12}\eta_{\mu\nu}\eta_{\alpha\beta}\partial^4+\frac{1}{6}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta-\frac{1}{2}\eta_{\mu\alpha}\partial_\beta\partial_\nu+\\
&+\frac{1}{12}(\eta_{\mu\nu}\partial_\alpha\partial_\beta+\eta_{\alpha\beta}\partial_\mu\partial_\nu)\partial^2\Big]h^{\alpha\beta}+\cdots\,, \qquad (2.9)
\end{aligned}
$$

where the dots stand for the lower and higher order terms.

Taking into account the formulas

$$\begin{aligned}\Gamma(2-\omega) &= \frac{1}{2-\omega}+\mathcal{O}(2-\omega)\,,\\ \Gamma(1-\omega) &= -\frac{1}{2-\omega}-1+\mathcal{O}(2-\omega)\,,\\ \Gamma(-\omega) &= \frac{1}{2(2-\omega)}+\frac{3}{4}+\mathcal{O}(2-\omega)\,,\end{aligned}\tag{2.10}$$

one can easily identify the divergent part of the polarization operator (sum of the two expressions (2.4) and (2.5)), with the expansion (2.6)-(2.9) of the divergent part of the effective action

$$\begin{aligned}\bar{\Gamma}^{(1)}_{div}(diagr) = -\frac{1}{2(4\pi)^2(\omega-2)}\int d^4xg^{1/2}\Big\{&\frac{1}{60}W+\frac{1}{2}\Big(\xi-\frac{1}{6}\Big)^2R^2+\\ &+\Big(\xi-\frac{1}{6}\Big)m^2R+\frac{1}{2}m^4\Big\}\,,\end{aligned}\tag{2.11}$$

that is almost the same as eq. (1.7). The main difference is that the expression (2.11) does not contain surface terms $\int d^4xg^{1/2}E$ and $\int d^4xg^{1/2}\nabla^2R$ which do not contribute to the propagator.

After subtracting divergences (together with the $\ln[4\pi\mu^2/m^2]$ terms) and neglecting the $\mathcal{O}(\omega-2)$ terms, the polarization operator becomes

$$\begin{aligned}\Pi^{\mu\nu,\alpha\beta}(p) = \frac{1}{2(4\pi)^2}\Bigg\{&\frac{1}{2}\delta^{\mu\nu,\alpha\beta}\Big[p^4C_3-2p^2m^2C_2+m^4C_1+\frac{p^4}{40}+\frac{p^2m^2}{4}+\frac{3m^4}{4}\Big]+\\ &+\frac{1}{4}\eta^{\mu\nu}\eta^{\alpha\beta}\Big[p^4(C_1+5C_2+C_3)-p^2m^2(3C_1+2C_2)+\\ &\qquad+m^4C_1-\frac{9p^4}{40}-\frac{p^2m^2}{4}-\frac{3m^4}{4}\Big]+\\ &+\frac{1}{2}(p^\mu p^\nu\eta^{\alpha\beta}+p^\alpha p^\beta\eta^{\mu\nu})\Big[p^2(C_2+2C_3)-2m^2C_2+\frac{p^2}{30}+\frac{m^2}{6}\Big]+\\ &+2C_3p^\mu p^\nu p^\alpha p^\beta+\frac{1}{4}(\eta^{\mu\alpha}p^\nu p^\beta+\eta^{\mu\beta}p^\nu p^\alpha+\eta^{\nu\alpha}p^\mu p^\beta+\eta^{\nu\beta}p^\mu p^\alpha)\times\\ &\times\Big[p^2(C_2+4C_3)-m^2(C_1+4C_2)-\frac{p^2}{60}-\frac{m^2}{6}\Big]-\\ &-2\xi^2C_1(p^\mu p^\nu-p^2\eta^{\mu\nu})(p^\alpha p^\beta-p^2\eta^{\alpha\beta})-\\ &-\frac{\xi}{2}(p^\mu p^\nu-p^2\eta^{\mu\nu})\Big[\Big(C_1p^2-2C_1m^2+2p^2C_2-\frac{p^2}{6}\Big)\eta^{\alpha\beta}+4p^\alpha p^\beta C_2\Big]-\\ &-\frac{\xi}{2}(p^\alpha p^\beta-p^2\eta^{\alpha\beta})\Big[\Big(C_1p^2-2C_1m^2+2p^2C_2-\frac{p^2}{6}\Big)\eta^{\mu\nu}+4p^\mu p^\nu C_2\Big]+\\ &+\frac{\xi m^2}{2}\Big[-p^2\delta^{\mu\nu,\alpha\beta}+p^2\eta^{\mu\nu}\eta^{\alpha\beta}-(p^\mu p^\nu\eta^{\alpha\beta}+p^\alpha p^\beta\eta^{\mu\nu})+\\ &\qquad+\frac{1}{2}\Big(\eta^{\mu\alpha}p^\nu p^\beta+\eta^{\mu\beta}p^\nu p^\alpha+\eta^{\nu\alpha}p^\mu p^\beta+\eta^{\nu\beta}p^\mu p^\alpha\Big)\Big]\Bigg\}\,,\end{aligned}\tag{2.12}$$

where

$$
\begin{aligned}
C_1 &= -\int_0^{1/2} dy \ln\left[\frac{p^2}{m^2}\Big(\frac{1}{4}-y^2\Big)+1\right], \\
C_2 &= \int_0^{1/2} dy \left(\frac{1}{4}-y^2\right) \ln\left[\frac{p^2}{m^2}\left(\frac{1}{4}-y^2\right)+1\right], \\
C_3 &= -\int_0^{1/2} dy \left(\frac{1}{4}-y^2\right)^2 \ln\left[\frac{p^2}{m^2}\left(\frac{1}{4}-y^2\right)+1\right]
\end{aligned} \tag{2.13}
$$

are functions of the ratio p^2/m^2, which can be easily calculated

$$
\begin{aligned}
C_1 &= A\,, \qquad C_2 = \frac{A}{36a^2} - \frac{A}{4} + \frac{1}{36}\,, \\
C_3 &= \frac{A}{5a^4} - \frac{A}{6a^2} + \frac{A}{16} + \frac{1}{60a^2} - \frac{41}{3600}\,,
\end{aligned}
$$

where

$$
A = 1 - \frac{1}{a}\ln\left(\frac{2+a}{2-a}\right), \qquad a^2 = \frac{4p^2}{p^2+4m^2}\,. \tag{2.14}
$$

For the terms without ξ we can compare the expression (2.12) and the corresponding covariant expression

$$
S^R_{vac} = \int d^4x g^{1/2}\left\{-\theta_G R - \theta_\Lambda + \theta_1 C^2 + \theta_2 R^2\right\}. \tag{2.15}
$$

Using the bilinear expansions (2.6), (2.7), (2.8), (2.9), we arrive at the following relations for θ_1, θ_2, θ_G, θ_Λ in the momentum space (the expression (2.15) must be understood such that all θ's act as operators in the bilinear expansions of the corresponding covariant terms):

$$
\begin{aligned}
\frac{\theta_1 p^4}{2} + \frac{\theta_G p^2}{4} - \frac{\theta_\Lambda}{4} &= -\frac{1}{4(4\pi)^2}\left[\frac{C_3 p^4}{2} - p^2 m^2 C_2 + \frac{C_1 m^4}{2} + \frac{p^4}{80} + \frac{p^2m^2}{8} + \frac{3m^4}{8}\right], \\
-\frac{\theta_1 p^4}{3} + 2\theta_2 p^4 - \frac{\theta_G p^2}{2} + \frac{\theta_\Lambda}{4} &= -\frac{1}{4(4\pi)^2}\left[\frac{(C_1+5C_2+C_3)p^4}{2} - \frac{p^2m^2(3C_1+2C_2)}{2} + \right. \\
&\qquad \left. + \frac{m^4 C_1}{2} - \frac{9p^4}{80} - \frac{p^2m^2}{8} - \frac{3m^4}{8}\right], \\
\left(\frac{\theta_1}{3} - 2\theta_2\right)p^2 + \frac{\theta_G}{2} &= -\frac{1}{4(4\pi)^2}\left[(2C_3+C_2)p^2 - 2m^2 C_2 + \frac{p^2}{30} + \frac{m^2}{6}\right], \\
-\frac{\theta_1 p^2}{2} - \frac{\theta_G}{4} &= -\frac{1}{4(4\pi)^2}\left[\left(2C_3 + \frac{C_2}{2}\right)p^2 - \left(2C_2 + \frac{C_1}{2}\right)m^2 - \frac{p^2}{120} - \frac{m^2}{12}\right], \\
\frac{2\theta_1}{3} + 2\theta_2 &= -\frac{C_3}{(4\pi)^2}\,.
\end{aligned} \tag{2.16}
$$

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Furthermore, it is easy to check that the ξ-dependent terms can be cast into the form

$$
\begin{aligned}
-\frac{\xi}{(4\pi)^2}\Bigg\{ &(p^\mu p^\nu - p^2\eta^{\mu\nu})\left(p^\alpha p^\beta - p^2\eta^{\alpha\beta}\right)\left[2\xi A + \frac{4A}{3a^2} - A + \frac{1}{9}\right] + \\
&+ \frac{m^2}{4}\Bigg[p^2\delta^{\mu\nu,\alpha\beta} - p^2\eta^{\mu\nu}\eta^{\alpha\beta} + \left(p^\mu p^\nu\eta^{\alpha\beta} + p^\alpha p^\beta\eta^{\mu\nu}\right) + \\
&- \frac{1}{2}\left(\eta^{\mu\alpha}p^\nu p^\beta + \eta^{\mu\beta}p^\nu p^\alpha + \eta^{\nu\alpha}p^\mu p^\beta + \eta^{\nu\beta}p^\mu p^\alpha\right)\Bigg]\Bigg\} .
\end{aligned}
\tag{2.17}
$$

This can be easily identified with eq. (2.8) and (2.7), therefore the ξ-dependent terms are associated with the $\int R^2$ and $\int m^2 R$-terms in the effective action and contribute to the θ_2 and θ_G coefficients only.

Despite the number of the equations in (2.16) exceeds the number of the unknowns, all these equations can be indeed satisfied. Taking into account the ξ-dependent terms in θ_2, we obtain

$$
\begin{aligned}
\theta_\Lambda &= \frac{3m^4}{8(4\pi)^2} , \\
\theta_G &= \frac{m^2}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right) , \\
\theta_1 &= -\frac{1}{4(4\pi)^2}\left(\frac{8A}{15a^4} + \frac{2}{45a^2}\right) , \\
\theta_2 &= \frac{A}{4(4\pi)^2}\left[- 2\left(\xi - \frac{1}{6}\right)^2 + \frac{1}{3}\left(\xi - \frac{1}{6}\right)\left(1 - \frac{4}{a^2}\right) - \frac{2}{9a^4} + \frac{1}{9a^2} - \frac{1}{72}\right] - \frac{1}{216(4\pi)^2 a^2} .
\end{aligned}
\tag{2.18}
$$

The coefficients (2.18) contain all information about the polarization operator. They will be checked, using the covariant Avramidi and Barvinsky-Vilkovisky technique in the next section. After that we shall consider the renormalization group for the coefficients of the vacuum action (1.1).

3. The covariant derivation of the effective action

The calculation of the polarization operator, presented in section 2, is rather complicated, so it is useful to verify it using a covariant method and compare the two results. To this end, we need to obtain the covariant effective action up to the second order in curvature. The one-loop effective action can be presented as the functional trace of the proper-time integral of the heat kernel $K(s)$. This representation of the effective action has been obtained by Avramidi using summation of the Schwinger-DeWitt series [14] and by Barvinsky and Vilkovisky using generalized Schwinger-DeWitt technique [15].

In the case of a massive scalar field, the formulas of [15] must be slightly modified and the one-loop euclidean effective action (up to the second order in curvature) reads (compare to the formulas (1.8) and (2.1) of [15])

$$
\bar{\Gamma}^{(1)} = -\frac{1}{2}\mathrm{Tr}\ln\left(-\nabla^2\hat{1} + m^2 - \hat{P} + \frac{1}{6}R\hat{1}\right) = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\mathrm{tr}K(s) ,
\tag{3.1}
$$

where $\hat{P} = -(\xi - 1/6)R$ and

$$\begin{aligned} \mathrm{tr}\, K(s) &= \frac{(\mu^2)^{2-w}}{(4\pi s)^w} \int d^4x g^{1/2} e^{-sm^2} \times \\ &\times \mathrm{tr}\Big\{ \hat{1} + s\hat{P} + s^2 \Big[R_{\mu\nu} f_1(-s\nabla^2) R^{\mu\nu} + R f_2(-s\nabla^2) R + \hat{P} f_3(-s\nabla^2) R + \\ &\qquad + \hat{P} f_4(-s\nabla^2)\hat{P} \Big] \Big\} + \mathcal{O}(\mathcal{R}^3)\,, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} f_1(\tau) &= \frac{f(\tau) - 1 + \tau/6}{\tau^2}\,, \qquad f_2(\tau) = \frac{f(\tau)}{288} + \frac{f(\tau) - 1}{24\tau} - \frac{f(\tau) - 1 + \tau/6}{8\tau^2}\,, \\ f_3(\tau) &= \frac{f(\tau)}{12} + \frac{f(\tau) - 1}{2\tau}\,, \qquad f_4(\tau) = \frac{f(\tau)}{2} \end{aligned} \tag{3.3}$$

and

$$f(\tau) = \int_0^1 d\alpha e^{\alpha(1-\alpha)\tau}\,, \qquad \tau = -s\nabla^2\,. \tag{3.4}$$

The integral over the proper time s is divergent and must be regularized. Below we adopt the dimensional regularization in the form suggested by Brown and Cassidy [16] (see also [17] for useful technical details). Also, inserting the $\exp(-sm^2)$ factor into (3.2) enables one to study the UV limit $\tau/sm^2 \gg 1$ and the IR limit $\tau/sm^2 \ll 1$ instead of the limits $\tau \gg 1$ and $\tau \ll 1$ which were investigated in [15]. The interface between UV and IR limits is our main subject of interest here.

It proves useful to change the variable $sm^2 = t$ and also denote

$$u = \frac{\tau}{t} = -\frac{\nabla^2}{m^2}\,.$$

After simple calculations we arrive at the following representation for the effective action (3.1):

$$\begin{aligned} \bar{\Gamma}^{(1)} &= -\frac{1}{2(4\pi)^2} \int d^4x g^{1/2} \left(\frac{m^2}{4\pi\mu^2} \right)^{w-2} \int_0^\infty dt e^{-t} \times \\ &\times \left\{ \frac{m^4}{t^{w+1}} + \left(\xi - \frac{1}{6} \right) \frac{Rm^2}{t^w} + \sum_{i=1}^{3} l_i^* \cdot R_{\mu\nu} M_i R^{\mu\nu} + \sum_{j=1}^{5} l_j \cdot R M_j R \right\}, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} l_1^* &= 1\,, \qquad l_2^* = \frac{1}{6}\,, \qquad l_3^* = -1\,; \\ l_1 &= \frac{1}{288} - \frac{1}{12}\left(\xi - \frac{1}{6}\right) + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2, \qquad l_2 = \frac{1}{24} - \frac{1}{2}\left(\xi - \frac{1}{6}\right), \\ l_3 &= -\frac{1}{8}\,, \qquad l_4 = -\frac{1}{16} + \frac{1}{2}\left(\xi - \frac{1}{6}\right), \qquad l_5 = \frac{1}{8} \end{aligned} \tag{3.6}$$

and

$$M_1 = \frac{f(tu)}{u^2 t^{w+1}}\,, \quad M_2 = \frac{f(tu)}{u t^w}\,, \quad M_3 = \frac{f(tu)}{u^2 t^{w+1}}\,, \quad M_4 = \frac{1}{u t^w}\,, \quad M_5 = \frac{1}{u^2 t^{w+1}}\,. \tag{3.7}$$

The calculation of the integrals in (3.5) is quite tedious. Let us present just the result, using notations

$$A = -\frac{1}{2}\int_0^1 d\alpha \ln\Big[1+\alpha(1-\alpha)u\Big] = 1 - \frac{1}{a}\ln\frac{1+a/2}{1-a/2}\,, \qquad a^2 = \frac{4\nabla^2}{\nabla^2 - 4m^2} \tag{3.8}$$

(indeed, this is equivalent to the previous definition eq. (2.14)), relations (2.10) and the expansion (we omit, systematically, those terms which contribute only to $\mathcal{O}(w-2)$-terms)

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{w-2} = 1 + (2-w)\ln\left(\frac{4\pi\mu^2}{m^2}\right) + \cdots .$$

The effective action can be cast into the form

$$\begin{aligned}
\bar{\Gamma}^{(1)} = \frac{1}{2(4\pi)^2}\int d^4x g^{1/2}\Bigg\{ & \frac{m^4}{2}\cdot\left[\frac{1}{2-w} + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + \frac{3}{2}\right] + \\
& + \left(\xi - \frac{1}{6}\right) m^2 R\left[\frac{1}{2-w} + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + 1\right] + \\
& + \frac{1}{2}C_{\mu\nu\alpha\beta}\left[\frac{1}{60(2-w)} + \frac{1}{60}\ln\left(\frac{4\pi\mu^2}{m^2}\right) + k_W(a)\right] C^{\mu\nu\alpha\beta} + \\
& + R\left[\frac{1}{2}\left(\xi-\frac{1}{6}\right)^2\left(\frac{1}{2-w} + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right) + k_R(a)\right] R\Bigg\},
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
k_W(a) &= \frac{8A}{15\,a^4} + \frac{2}{45a^2} + \frac{1}{150}\,, \\
k_R(a) &= A\left(\xi-\frac{1}{6}\right)^2 - \frac{A}{6}\left(\xi-\frac{1}{6}\right) + \frac{2A}{3a^2}\left(\xi-\frac{1}{6}\right) + \frac{A}{9a^4} - \frac{A}{18a^2} + \frac{A}{144} + \\
&\quad + \frac{1}{108a^2} - \frac{7}{2160} + \frac{1}{18}\left(\xi-\frac{1}{6}\right)^2 .
\end{aligned} \tag{3.10}$$

In eq. (3.9) we used the relation $C^2 = E + 2W$, that was justified in [15], including the terms with non-local $1/\nabla^2$-type insertions.

It is easy to see that the above expression for the effective action perfectly corresponds to the polarization operator derived in the previous section. This coincidence represents reliable verification of the results and enables one to obtain robust conclusions concerning the renormalization group in curved space-time.

Another important check of the expression (3.9) is the following. In the massless limit $m \to 0$ we expect to meet the standard massless result, and in particular the anomaly-induced effective action. Taking $m \to 0$ we arrive at the singular expression for A in (3.8). On the other hand, the consistency of the conformal theory (and in particular the existence of the anomaly-induced effective action [18]) requires $\xi = 1/6$. After we set $\xi = 1/6$, the divergent A-dependent terms cancel, and in the $R[\ldots]R$ sector we meet only the local term

$$\bar{\Gamma}^{(1)}\left(\xi = \frac{1}{6}, m \to 0\right) = -\frac{1}{12\cdot 180(4\pi)^2}\int d^4x g^{1/2} R^2 + \ldots . \tag{3.11}$$

Taking into account the well-known relation

$$\frac{2}{g^{1/2}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^4x g^{1/2} R^2 = 12 \nabla^2 R \,,$$

we obtain the standard expression for the $\nabla^2 R$-term in the conformal anomaly

$$\langle T^\mu_\mu \rangle = -\frac{1}{180(4\pi)^2} \nabla^2 R + \dots \,, \qquad (3.12)$$

where the dots stand for the E and C^2 terms which we do not discuss here. Let us remark that this test of the calculations is rather complete, because the coefficient in (3.12) depends on almost all terms in the expression for the effective action. Those terms which do not contribute to (3.12) directly, are verified through the cancelation of the singular (at $m \to 0$) structures.

4. Renormalization group equations

In the $\overline{\text{MS}}$ scheme the β-function of the effective charge C is defined as

$$\beta_C \left(\overline{\text{MS}}\right) = \lim_{n \to 4} \mu \frac{dC}{d\mu} \,. \qquad (4.1)$$

It is easy to see that when this procedure is applied to the expressions (3.9) or (2.18), the $\beta(\overline{\text{MS}})$-functions for the parameters Λ/G, $, 1/G$, $a_{1,2}$ are exactly the same as the usual ones, obtained in the covariant approach (1.9) and (1.10).

The disadvantage of the $\overline{\text{MS}}$ scheme is that one can not control the decoupling of the massive quantum fields in an external gravitational field, hence one has to go beyond this scheme. The explicit calculations of the one-loop diagrams of massive fields presented in the previous sections enable one to apply the physical mass-dependent scheme of renormalization and raise a hope to observe the decoupling. Unfortunately, this approach is not universal, because the calculations have been performed through the perturbative expansion of the metric on the flat background. Also, the covariant calculation in section 3 has been performed in the second order in curvature approximation and can not provide an information beyond that obtained in section 2. Therefore, one can not be sure to obtain a reliable information about the non-perturbative effects. At the same time, we can always use the correspondence with the standard results in the $\overline{\text{MS}}$ scheme in the UV regime as a criterion of correctness of the results obtained in the mass-dependent scheme.

The derivation of the β-functions in the mass-dependent scheme has been described, e.g. in [13]. Starting from the polarization operator, one has to subtract the counterterm at the momentum $p^2 = M^2$, where M is the renormalization point. Then, the β-function is defined (instead of eq. (4.1)) as

$$\beta_C = \lim_{n \to 4} M \frac{dC}{dM} \,. \qquad (4.2)$$

Mathematically, this is equivalent to taking the derivative $-pd/dp$ of the formfactors in the polarization operator. If we apply this procedure to the formfactor of the $C^2_{\mu\nu\alpha\beta}$-term, the result for the β-function in a mass-dependent scheme is

$$\beta_1 = -\frac{1}{(4\pi)^2}\left(\frac{1}{18a^2} - \frac{1}{180} - \frac{a^2-4}{6a^4}A\right), \tag{4.3}$$

that is the general result for the one-loop β-function valid at any scale. As the special cases we meet the UV limit $p^2 \gg m^2$

$$\beta_1^{UV} = -\frac{1}{(4\pi)^2}\frac{1}{120} + \mathcal{O}\left(\frac{m^2}{p^2}\right), \tag{4.4}$$

that agrees with the $\overline{\text{MS}}$-scheme result (1.10) and also the IR limit $p^2 \ll m^2$:

$$\beta_1^{IR} = -\frac{1}{1680(4\pi)^2}\cdot\frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right). \tag{4.5}$$

Similar calculations for the coefficient of the R^2-term give

$$\begin{aligned}\beta_2 = \lim_{n\to 4} M\frac{da_2}{dM} = -\frac{1}{(4\pi)^2}\Bigg\{&\frac{1}{8}(4A - a^2A + a^2)\left(\xi - \frac{1}{6}\right)^2 - \frac{a^2-4}{48}\left(\xi - \frac{1}{6}\right) + \\ &+ \frac{(a^2-4)(a^2-12)A}{48a^2}\left(\xi - \frac{1}{6}\right) + \\ &+ \frac{a^2-4}{8}\left[\frac{A}{6a^2} - \frac{A}{144} - \frac{5A}{9a^4} + \frac{1}{144} - \frac{5}{108a^2}\right]\Bigg\}. \end{aligned}\tag{4.6}$$

In the UV limit $p^2 \gg m^2$ the β-function is (in agreement with the standard result (1.10))

$$\beta_2^{UV} = -\frac{1}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right)^2 + \mathcal{O}\left(\frac{m^2}{p^2}\right). \tag{4.7}$$

while in the IR limit $p^2 \ll m^2$ we obtain:

$$\beta_2^{IR} = -\frac{1}{12(4\pi)^2}\left[\left(\xi - \frac{1}{6}\right)^2 - \frac{1}{15}\left(\xi - \frac{1}{6}\right) + \frac{1}{630}\right]\cdot\frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right). \tag{4.8}$$

The expressions (4.3)–(4.8) are the main results of our work. In particular, the interpolation between the β-functions in the high-energy regime (4.4), (4.7) and the β-functions in the low-energy regime (4.5), (4.8) exhibits the decoupling phenomenon in the vacuum quantum effects of the massive scalar particle. Let us remark that the correctness of the calculations of the β-functions is verified by: i) precise correspondence in the UV with the known expressions obtained in the $\overline{\text{MS}}$ scheme; ii) great amount of cancelations of the singular $\mathcal{O}(m^4/p^4)$, $\mathcal{O}(m^2/p^2)$ and constant $\mathcal{O}(1)$-terms in the expressions for $\beta_{1,2}(IR)$ in (4.5) and (4.8); iii) correct result for the anomalous $\int R^2$-term in the massless limit. We remark that, due to covariance of the effective action, the expressions for the β-functions (4.3) and (4.6) should apply not only to the 2-point function but also to the vertex corrections.

At the same time, taking the derivative with respect to momenta of those terms in the effective action, which correspond to the cosmological constant and Einstein-Hilbert term, brings zero result. Therefore, at this level we do not have an explicit interpretation for the renormalization group equations for $1/G$ and Λ/G. Unfortunately, these β-functions can not be calculated in the mass-dependent scheme through the perturbative expansion of the metric on flat background.

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5. Discussions and conclusions

Making the perturbative calculations in a physical mass-dependent scheme, we arrived at the explicit expressions for the β-functions for the coefficients of the higher-derivative terms a_1 and a_2. The decoupling of massive degrees of freedom proceeds according to the rule

$$\beta(IR) \sim const \cdot \left(\frac{p^2}{m^2}\right), \qquad p^2 < m^2\,, \tag{5.1}$$

which has been guessed in [19] for the cosmological constant by analogy with a similar phenomena in QED. However, we have discovered this law not for the cosmological constant but only for the higher derivative terms.

As we have seen above, the perturbative calculations in the mass-dependent scheme do not reveal the β-functions in the case of $1/G$ and Λ/G. In these sectors the UV limit of the mass-dependent scheme differs from that in the $\overline{\mathrm{MS}}$ scheme. The question is whether these β-functions are really zero? From our point of view, this is not so. The reason for the difference between two renormalization schemes is that actual calculations has been performed on the flat background, while the $\overline{\mathrm{MS}}$-based derivation is completely covariant.

After all, there remains a possibility to have the scale dependence in the observables different from the polarization operator on the flat background. If this dependence really exists, it would be a non-perturbative effect with respect to the expansion (2.1), since the perturbative effects are controlled by general covariance. It is natural to expect that these non-perturbative effects really take place, for otherwise there would not be correspondence between the mass-dependent and $\overline{\mathrm{MS}}$ schemes in the high-energy regime. However, it is not for sure that the decoupling law for these non-perturbative effects will be the same as for the perturbatively accessible sectors such as the higher-derivative parameters $a_{1,2}$.

It is interesting to analyze further the reasons why the "perturbative" β_Λ and $\beta_{1/G}$ are not seen in the mass-dependent scheme. In the perturbative approach, as for any Quantum Field Theory in flat space-time, the β-functions reflect the momentum-dependence of the formfactors in the polarization operator. In the covariant formalism, this dependence corresponds to the non-local terms in the effective action of vacuum.

We find it instructive once again to consider the QED example. Due to gauge invariance, momentum running of effective coupling constant in QED is connected with renormalization of the photon propagator (the Z_3 renormalization constant) and the effective action has the following non-local contribution (see, e.g. [21]) in dimensional regularization for $p^2 \gg m_e^2$,

$$S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2}{3(4\pi)^2}F_{\mu\nu}\ln\left(-\frac{\nabla^2}{\mu^2}\right)F^{\mu\nu}. \tag{5.2}$$

This action can be regarded as a non-local version of the well-known Euler-Heisenberg lagrangian [22]. Clearly, $\ln(-\nabla^2/\mu^2)$ corresponds to $\ln(p^2/\mu^2)$ contribution in momentum space, therefore, the electric charge 'runs' as the momentum changes. Similarly in gravity, as follows from our eq. (3.9) in section 3 for finite part there is the following non-local contributions to the euclidean action:

$$S = \frac{1}{2(4\pi)^2}\int d^4x g^{1/2}\left[\frac{1}{2}C_{\mu\nu\alpha\beta}k_W(a)C^{\mu\nu\alpha\beta} + Rk_R(a)R\right], \tag{5.3}$$

where the functions k_W and k_R are given by eq. (3.10). At high energy limit we find the familiar $\ln(-\nabla^2/m^2)$-type contribution (see, e.g. [20] where the high-energy limit has been restored from the general covariant $\overline{\rm MS}$ renormalization group, and also [21])

$$k_W \approx -\frac{1}{60}\ln\left(-\frac{\nabla^2}{m^2}\right) \qquad \text{and} \qquad k_R \approx -\frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 \ln\left(-\frac{\nabla^2}{m^2}\right). \tag{5.4}$$

At low energy massive particles decouple and k_W and k_R can be formally expanded in series in ∇^2/m^2 to give

$$k_W \approx -\frac{1}{840}\left(-\frac{\nabla^2}{m^2}\right),$$

and

$$k_R \approx -\frac{1}{12}\left[\left(\xi - \frac{1}{6}\right)^2 - \frac{1}{15}\left(\xi - \frac{1}{6}\right) + \frac{1}{630}\right]\left(-\frac{\nabla^2}{m^2}\right). \tag{5.5}$$

Thus, the IR running of the coupling constants a_1 and a_2 is quite similar to that of QED. However, for the cosmological constant and the Einstein term, the situation is different. The reason is simple. It is clear that unlike the C^2 and R^2 terms (these two are similar to the $F_{\mu\nu}F^{\mu\nu}$ term in the case of QED) it is impossible to ensure renormalization group running for the cosmological constant and Einstein term by inserting a certain function of the operator ∇^2/m^2 into the action terms because this operation produces either zero (for the cosmological constant) or complete derivative (for the Einstein term).

According to [4, 5, 6, 8], the renormalization group scaling in curved spacetime is defined as scaling of the metric $g_{\mu\nu} \to e^{2t}g_{\mu\nu}$. This is the key idea of renormalization group in curved spacetime because momentum is not defined in a general curved spacetime and, hence, the familiar flat space momentum scaling $p \to e^{-t}p$ cannot be used. The operator $\nabla^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu$ scales like $\nabla^2 \to e^{-2t}\nabla^2$ that corresponds to the flat spacetime scaling $p^2 \to e^{-2t}p^2$. If the running for the cosmological constant and Einstein terms could be seen in the framework of the perturbative expansion, in the second order in $h_{\mu\nu}$ it can only be related with the terms

$$\int d^4x g^{1/2} R_{\mu\nu}\left(-\frac{m^2}{\nabla^2}\right)^2 R^{\mu\nu}, \qquad \int d^4x g^{1/2} R\left(-\frac{m^2}{\nabla^2}\right)^2 R$$

for the cosmological constant and with the expressions

$$\int d^4x g^{1/2} R_{\mu\nu}\left(-\frac{m^2}{\nabla^2}\right) R^{\mu\nu}, \qquad \int d^4x g^{1/2} R\left(-\frac{m^2}{\nabla^2}\right) R$$

for the Einstein-Hilbert term. Indeed, only these terms have an appropriate scaling, are of the second order in curvature, and admit the $\ln(-\nabla^2/m^2)$ insertion similar to that of (5.4).

Within the perturbative scheme, one can see the renormalization group running of those terms only, and not of the original cosmological and Einstein-Hilbert terms. But, our results show that the running for these two structures doesn't take place. Probably, this is related to the locality of the divergences in the effective action. Another obstacle for the appearance of these non-local terms arises in the IR limit, where the presence of ∇^2 in denominator would violate a nice IR limit of gravity [26, 12]. Indeed, the absence of the β-functions for $1/G$ and Λ/G looks as an artefact of the perturbative expansion in $h_{\mu\nu}$ rather than the fundamental property of the renormalization group in curved space-time. We expect that if the calculations were performed on a non-flat background, there would be nontrivial β-functions at high energy in a mass-dependent scheme. This would provide the correspondence with the covariant $\overline{\rm MS}$ result (1.9) at high energy and the information concerning the decoupling at low energy.

Another argument supporting the existence of the running for Λ and G comes from the derivation of the effective action in the covariant proper-time cut-off regularization (see appendix B), where we have found the standard β-functions for the cosmological and inverse Newton constants at the UV regime and the exponential decoupling of the massive fields contributions to these β-functions in the IR regime (unfortunately, the IR limit in the covariant proper-time cut-off regularization is not reliable).

In conclusion, we have performed the calculations in the mass-dependent renormalization scheme and found the explicit law for the decoupling of the massive degrees of freedom for the higher derivative terms in the vacuum effective action. Only more complicated calculations on the non-flat background can provide a reliable information about the renormalization group for the cosmological and inverse Newton constants.

The form of the IR decoupling rule in the higher derivative sector indicates to the possibility of the soft transition between the stable and unstable regimes in the anomaly-induced inflation [1, 23]. The detailed investigation of this application will be reported separately.

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Note added. After this paper has been submitted, we learned about the work by Bastianelli and Zirotti [28], which is also devoted to the derivation of quantum corrections for the massive non-minimal scalar. The calculations of [28] are based on the worldline formalism, and their final result corresponds to our intermediate expressions (2.4), (2.5) before the integrals from the appendix A are taken. Technically, this integration and the consequent identification with covariant terms in the gravitational action are the most difficult parts of our work. However, the correspondence of the intermediate formulas provides one more useful verification of the calculations.

A. The table of useful integrals

In section 2 we have used dimensional regularization. The following integrals were relevant:

$$I_1 = \int \frac{d^{2w}k}{(2\pi)^{2w}} \frac{1}{(k^2+m^2)([k-p]^2+m^2)}$$

$$= \frac{\Gamma(2-w)}{(4\pi)^w}(m^2)^{w-2}\,{}_2F_1\left(2-w,1,\frac{3}{2};\frac{-p^2}{4m^2}\right), \quad \text{(A.1)}$$

$$\int \frac{d^{2w}k}{(2\pi)^{2w}} \frac{k^\mu}{(k^2+m^2)([k-p]^2+m^2)} = \frac{I_1}{2}p^\mu = p^\mu I_2\,, \quad \text{(A.2)}$$

$$\int \frac{d^{2w}k}{(2\pi)^{2w}} \frac{k^\mu k^\nu}{(k^2+m^2)([k-p]^2+m^2)} = I_3\eta^{\mu\nu} + I_4 p^\mu p^\nu\,, \quad \text{(A.3)}$$

$$I_3 = \frac{\Gamma(1-w)}{2(4\pi)^w}(m^2)^{w-1}\,{}_2F_1\left(1-w,1,\frac{3}{2};\frac{-p^2}{4m^2}\right), \quad \text{(A.4)}$$

$$I_4 = \frac{\Gamma(2-w)}{2(4\pi)^w}(m^2)^{w-2}\times$$

$$\times\left[{}_2F_1\left(2-w,1,\frac{3}{2};\frac{-p^2}{4m^2}\right) - \right.$$

$$\left. -\frac{1}{3}\,{}_2F_1\left(2-w,2,\frac{5}{2};\frac{-p^2}{4m^2}\right)\right], \quad \text{(A.5)}$$

$$\int \frac{d^{2w}k}{(2\pi)^{2w}} \frac{k^\mu k^\nu k^\alpha}{(k^2+m^2)([k-p]^2+m^2)} = I_5 p^\mu p^\nu p^\alpha + 3p^{(\alpha}\eta^{\mu\nu)} I_6\,, \quad \text{(A.6)}$$

where the parenthesis indicate to the symmetrization (e.g. $D^{(\alpha\beta)} = 1/2[D^{\alpha\beta} + D^{\beta\alpha}]$),

$$I_5 = \frac{\Gamma(2-w)}{2(4\pi)^w}(m^2)^{w-1}\left[{}_2F_1\left(2-w,1,\frac{3}{2};\frac{-p^2}{4m^2}\right) - \frac{1}{2}\,{}_2F_1\left(2-w,2,\frac{5}{2};\frac{-p^2}{4m^2}\right)\right], \quad \text{(A.7)}$$

$$I_6 = \frac{\Gamma(1-w)}{4(4\pi)^w}(m^2)^{w-2}\,{}_2F_1\left(1-w,1,\frac{3}{2};\frac{-p^2}{4m^2}\right), \quad \text{(A.8)}$$

$$\int \frac{d^{2w}k}{(2\pi)^{2w}} \frac{k^\mu k^\nu k^\alpha k^\beta}{(k^2+m^2)([k-p]^2+m^2)} = \quad \text{(A.9)}$$

$$= I_7 p^\mu p^\nu p^\alpha p^\beta + 6p^{(\alpha}\eta^{\mu\nu}p^{\beta)} I_8 + 3\eta^{(\alpha\beta}\eta^{\mu\nu)} I_9\,, \quad \text{(A.10)}$$

$$I_7 = \frac{\Gamma(2-w)}{(4\pi)^w}(m^2)^{w-2}\left[\frac{1}{2}\,{}_2F_1\left(2-w,1,\frac{3}{2};\frac{-p^2}{4m^2}\right) - \frac{1}{3}\,{}_2F_1\left(2-w,2,\frac{5}{2};\frac{-p^2}{4m^2}\right) + \right.$$

$$\left. + \frac{1}{30}\,{}_2F_1\left(2-w,3,\frac{7}{2};\frac{-p^2}{4m^2}\right)\right], \quad \text{(A.11)}$$

$$I_8 = \frac{\Gamma(1-w)}{4(4\pi)^w}(m^2)^{w-1}\left[{}_2F_1\left(1-w,1,\frac{3}{2};\frac{-p^2}{4m^2}\right) - \frac{1}{3}\,{}_2F_1\left(1-w,2,\frac{5}{2};\frac{-p^2}{4m^2}\right)\right], \quad \text{(A.12)}$$

$$I_9 = \frac{\Gamma(-w)}{4(4\pi)^w}(m^2)^{w}\,{}_2F_1\left(-w,1,\frac{3}{2};\frac{-p^2}{4m^2}\right). \quad \text{(A.13)}$$

Here,

$${}_2F_1\Big(a,b,c;z\Big) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad \text{(A.14)}$$

is the hypergeometric function of z and

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

is the Pochhammer symbol.

B. The covariant proper-time cut-off regularization

In order to understand better our result concerning the β-functions, let us perform an alternative covariant calculation using the cut-off regularization of the proper-time integral in the Schwinger-De Witt approach. We shall use the same proper-time integral and the expression for the heat-kernel as in the section 3. In some sense, the use of the covariant cut-off regularization provides a small advantage, because one can investigate not only logarithmic, but also quadratic and quartic divergences.

In the covariant cut-off regularization the effective action is defined as

$$\bar{\Gamma}^{(1)} = -\frac{1}{2}\int\limits_{1/\Omega^2}^{\infty}\frac{ds}{s}\mathrm{tr}K(s)\,, \tag{B.1}$$

where Ω is a cut-off parameter with the dimension of mass and all other notations are the same as in eq. (3.2). Let us perform the analysis separately for the cosmological constant, Einstein-Hilbert terms and for the higher-derivative terms.

(i) For the cosmological constant term, using the same change of variables as in section 3, we obtain

$$\Gamma^{(1)}_{\Lambda} = -\frac{1}{2}\int\limits_{1/\Omega^2}^{\infty}\frac{ds}{s}e^{-m^2 s}\frac{1}{(4\pi s)^2} = -\frac{1}{2(4\pi)^2}\int\limits_{m^2/\Omega^2}^{\infty}dtt^{-3}e^{-t} = -\frac{m^4}{2(4\pi)^2}\Gamma\Big(-2,\frac{m^2}{\Omega^2}\Big)\,, \tag{B.2}$$

where

$$\Gamma(\alpha,x) = \int\limits_{x}^{\infty}dte^{-t}t^{\alpha-1} \tag{B.3}$$

is the incomplete gamma function (see, e.g. [25], formula 8.350.2). Using the expansion

$$\Gamma(-n,x) = \frac{(-1)^n}{n!}\left[\Gamma(0,x) - e^{-x}\sum_{m=0}^{n-1}\frac{(-1)^m m!}{x^{m+1}}\right], \tag{B.4}$$

where $\Gamma(0,x) = -\mathrm{Ei}(-x)$ and

$$\mathrm{Ei}(-x) = C + \ln(x) + \sum_{k=1}^{\infty}\frac{(-x)^k}{k\cdot k!}\,, \qquad (x>0)\,, \tag{B.5}$$

for the large cut-off limit $\Omega^2 \gg m^2$ we find

$$\Gamma^{(1)}_{\Lambda} \approx -\frac{1}{4(4\pi)^2}\left[\Omega^4 + 2\Omega^2 m^2 + m^4\ln\left(\frac{\Omega^2}{m^2}\right)\right] + \dots . \tag{B.6}$$

The structure of divergences is the standard one. The divergences can be canceled by the corresponding counterterms. Let us choose them in the form

$$\Delta S = +\frac{1}{4(4\pi)^2}\left[\left(\Omega^4 + k_1\mu^4\right) + 2\left(\Omega^2 + k_2\mu^2\right)m^2 + m^4\ln\left(\frac{\Omega^2}{\mu^2}\right)\right]. \tag{B.7}$$

If the normalization condition is chosen in a simplest and natural way $k_1 = k_2 = 0$, the finite part of the effective action will not depend on the quartic and quadratic divergences (see also discussion in [3]). It is easy to see that the β_Λ can be calculated using the formula (1.9) and we arrive at the usual $\overline{\text{MS}}$-scheme renormalization group equation for the cosmological constant. Indeed, the same β-function can be obtained by taking the logarithmic derivative of the corresponding term in (B.6) with respect to the covariant cut-off $\Omega d/d\Omega$ with negative sign. Below we shall neglect the non-logarithmic divergences and always define β-functions by taking the last type of derivative.

In order to see the decoupling of the massive field at low energy, one can try to consider another extreme case $\Omega^2 \ll m^2$. The solution (B.2) gives, after we apply to it the derivative $\Omega(d/d\Omega)$

$$\beta_\Lambda^{IR} = -\frac{\Omega^4}{(4\pi)^2}e^{-m^2/\Omega^2} \tag{B.8}$$

Another way of getting the same formula is to use the asymptotics of the incomplete gamma function (see [25], formula 8.357) for large $|x|$

$$\Gamma(\alpha, x) = x^{\alpha-1}e^{-x}\left[\sum_{m=0}^{M-1}\frac{(-1)^m\Gamma(1-\alpha+m)}{x^m\Gamma(1-\alpha)} + \mathcal{O}\left(|x|^{-M}\right)\right], \qquad |x|\to\infty \tag{B.9}$$

and then apply the derivative $\Omega(d/d\Omega)$ to the expression

$$-\frac{m^4}{2(4\pi)^2}\Gamma\Big(-2, \frac{m^2}{\Omega^2}\Big) \approx -\frac{m^4}{2(4\pi)^2}\frac{\Omega^6}{m^6}e^{-m^2/\Omega^2}. \tag{B.10}$$

The expression (B.8) can be understood as a very fast decrease of β_Λ in the region below the natural scale m^2. It is tentative to consider eq. (B.10) as a hint supporting the sharp cut-off decoupling for the cosmological constant [3], in contrast to the hypothesis $\beta_\Lambda(IR) \sim \mu^2/m^2 \cdot \beta_\Lambda(UV)$ which has been suggested in [19]. Unfortunately, the physical sense of the cut-off in the gravitational setting and especially the mathematical consistency of the $\Omega^2 \ll m^2$ limit for the heat-kernel representation are not obvious. Moreover, for the higher derivative sector this definition is, as it will be shown below, contradictory. Finally, the decoupling in the lower-derivative sector of the gravitational action can not be understood by using the covariant proper-time cut-off, and we present it here just to complete the discussion.

(ii) The calculation of the divergences and the renormalization of the Einstein-Hilbert term goes the same way as for the cosmological constant term. The effective action is

$$\Gamma^{(1)}_{EH} = \frac{1}{2}\left(\xi - \frac{1}{6}\right)\int\limits_{1/\Omega^2}^{\infty}\frac{ds}{(4\pi s)^2}e^{-m^2 s}R$$

$$= \frac{m^2 R}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right)\int\limits_{m^2/\Omega^2}^{\infty}\frac{dt}{t^2}e^{-t} = \frac{m^2 R}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right)\Gamma\left(-1, \frac{m^2}{\Omega^2}\right). \tag{B.11}$$

For $\Omega^2 \gg m^2$ the result is

$$\Gamma^{(1)}_{EH} \approx \frac{1}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right) R\left[\Omega^2 - m^2 \ln\left(\frac{\Omega^2}{m^2}\right)\right] + \dots . \tag{B.12}$$

The form of the counterterms and the expression for the β-function is obvious.

$$\beta^{UV}_R = \frac{m^2}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right). \tag{B.13}$$

This result agrees with the previous one eq. (1.10) obtained in the $\overline{\text{MS}}$ scheme. In the low-energy limit $\Omega^2 \ll m^2$ we obtain

$$\beta^{IR}_R = -\frac{\Omega^2}{(4\pi)^2}e^{-m^2/\Omega^2}. \tag{B.14}$$

Of course, this is the same type of decoupling which we already met for the cosmological constant case. Unfortunately, this IR limit is inconsistent, as we shall see in brief.

(iii) For the higher-derivative sector we consider, for the sake of simplicity, only the $R_{\mu\nu}(..)R^{\mu\nu}$-type term. This term defines the renormalization of the C^2-terms, and therefore we can expect the correspondence with the expression (4.3) in both UV and IR limits. After the standard change of variables we arrive at the integral representation

$$\Gamma^{(1)}_{R^2_{\mu\nu}} = -\frac{1}{2(4\pi)^2}\int\limits_{m^2/\Omega^2}^{\infty}\frac{dt}{t}e^{-t}R_{\mu\nu}\left[\frac{1}{t^2u^2}\int_0^1 d\alpha e^{-\alpha(1-\alpha)tu} - \frac{1}{t^2u^2} + \frac{1}{6tu}\right]R^{\mu\nu}, \tag{B.15}$$

where, as before, $u = -\nabla^2/m^2$. After some algebra we arrive at the β-function

$$\beta_1 = -\frac{1}{2(4\pi)^2}e^{-m^2/\Omega^2}\left[\frac{\Omega^4}{\nabla^4}\int_0^1 d\alpha e^{\alpha(1-\alpha)\nabla^2/\Omega^2} - \frac{\Omega^4}{\nabla^4} - \frac{1}{6}\frac{\Omega^2}{\nabla^2}\right]. \tag{B.16}$$

At low energies we expand $\exp[\alpha(1-\alpha)\nabla^2/\Omega^2]$ into the series in ∇^2/Ω^2 and arrive at the expression

$$\beta_1 \approx -\frac{1}{120(4\pi)^2}e^{-m^2/\Omega^2}. \tag{B.17}$$

This expressions is in disagreement with the previous one eq. (4.5), which was obtained in the framework of the standard methods and the physical mass-dependent scheme. Hence, the IR limit $\Omega^2 \ll m^2$ of the renormalization group equations in the covariant cut-off approach is contradictory. In fact, this contradiction shows, once again, that the physical interpretation of the μ-dependence in gravity is quite complicated. It is possible that this interpretation depends on the concrete choice of the background metric and can not be formulated in the general form.

The source of the difference between the expressions (4.5) and (B.17) is obvious. The proper-time integral representation of the effective action is formally consistent if the integration performs from zero to infinity (see, e.g. [27]). Indeed, the lower limit of this integral is divergent and its UV regularization is related to the renormalization group. In particular, this can be seen from the UV limit $\Omega^2 \gg m^2$ which is consistent in all sectors. But, when taking the $\Omega^2 \ll m^2$ limit, we lose the link with the effective action and hence the corresponding results do not have much sense.

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